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A Convergence Theorem for Noncommutative Continued Fractions*

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We concern ourselves with the formal expression

$$\frac{A_1}{E+}\frac{A_2}{E+\cdots} \tag{1}$$

in which the A_k are members of a complex Banach algebra with identity E. Associated with (1) is the sequence

$$Q_n^{-1}P_n \tag{2}$$

(assuming Q_n to be invertible), where P_n and Q_n are given by

$$\begin{gathered} P_{n+1} = P_n + A_{n+1}P_{n-1}\,, \\ Q_{n+1} = Q_n + A_{n+1}Q_{n-1}\,, \\ P_1 = P_2 = A_1\,, \qquad Q_1 = E \qquad \text{and} \qquad Q_2 = E + A_2\,. \end{gathered} \tag{3}$$

The expression (1) converges if Q_n is invertible for sufficiently large n and (2) converges. In this case the value of (1) is defined to be the limit of the sequence (2).

We shall prove the following result which, in a sense, generalizes a theorem of Worpitzky [1] to our abstract setting.

THEOREM. *In* (1), *let* $\sup a_n < a < \frac{1}{4}$, *where* $a_n = \| A_n \|$. *Then* (1) *converges. Also, the constant* $\frac{1}{4}$ *is the best possible.*

Before proving the theorem, we record two lemmas dealing with ordinary (real) continued fractions. The proofs of the lemmas are straightforward and so are omitted.

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LEMMA 1. *If* $0 < a < \frac{1}{4}$, *then the convergents of the continued fraction*

$$\frac{a}{1-}\,\frac{a}{1-\cdots} \tag{4}$$

are positive, monotonic increasing, converging to

$$\tfrac{1}{2}[1-(1-4a)^{1/2}] < \tfrac{1}{2}(1-\delta), \tag{5}$$

for some $0 < \delta < 1$.

LEMMA 2. *Let* $b_k \geqslant 0$ *and* $\sup b_k < a < \frac{1}{4}$. *Then the n-th convergent of*

$$\frac{b_1}{1-}\,\frac{b_2}{1-\cdots} \tag{6}$$

is nonnegative and less than the n-th convergent of (4).

We set

$$F_n = A_n Q_{n-2} Q_{n-1}^{-1}, \qquad f_n = \| F_n \|, \qquad \text{and} \qquad q_n = \| Q_n^{-1} \|. \tag{7}$$

Proof of the Theorem. Obviously, Q_1 and Q_2 are invertible. Use of the lemmas gives

$$f_3 \leqslant a_3 \| (E + A_2)^{-1} \| \leqslant \frac{a_3}{1-a_2} < \frac{1-\delta}{2}, \qquad 0 < \delta < 1,$$

and

$$Q_3 = Q_2 + A_3 Q_1 = (E + F_3)\, Q_2,$$

implying the invertibility of Q_3. Inductively, assume that Q_k is invertible and that

$$f_k \leqslant \frac{a_k}{1-}\,\frac{a_{k-1}}{1-\cdots}\,\frac{a_3}{1-a_2} < \frac{1-\delta}{2}$$

for $k = 3, 4, \ldots, n$. Then by the properties of the norm and use of the lemmas,

$$\begin{aligned} f_{n+1} &\leqslant a_{n+1} \| Q_{n-1}[(E + F_n)\, Q_{n-1}]^{-1} \| \\ &= a_{n+1} \| (E + F_n)^{-1} \| \leqslant \frac{a_{n+1}}{1 - f_n} \leqslant \frac{a_{n+1}}{1-}\,\frac{a_n}{1-\cdots}\,\frac{a_3}{1-a_2} < \frac{1-\delta}{2}. \end{aligned}$$

Thus, since $Q_{n+1} = (E + F_{n+1})\, Q_n$, Q_{n+1} is invertible. So for all n, Q_n is invertible and

$$f_n < \frac{1-\delta}{2} < \frac{1}{2}. \tag{8}$$

Now $q_1 < 2$, $q_2 < 2$, and

$$q_{n+1} = \| Q_n^{-1}(E + F_{n+1})^{-1} \| \leqslant \frac{q_n}{1 - f_{n+1}} < 2q_n ,$$

and so by induction

$$q_n < 2^{n-1}. \tag{9}$$

Use of the identity

$$Q_{n+1}^{-1} P_{n+1} - Q_n^{-1} P_n = (-1)^n Q_{n+1}^{-1} F_{n+1} F_n \cdots F_3 A_2 A_1 \tag{10}$$

yields

$$\begin{aligned} \| Q_{n+1}^{-1} P_{n+1} - Q_n^{-1} P_n \| &\leqslant q_{n+1} f_{n+1} f_n \cdots f_3 a_2 a_1 \\ &< 2^{n-4} \left(\frac{1 - \delta}{2} \right)^{n-1} < (1 - \delta)^{n-1}, \end{aligned} \tag{11}$$

where $0 < \delta < 1$. Set $\rho = 1 - \delta$; then the triangle inequality along with (11) gives

$$\| Q_{n+k}^{-1} P_{n+k} - Q_n^{-1} P_n \| < \rho^{n+k-2} + \rho^{n+k-3} + \cdots + \rho^{n-1} < \frac{\rho^{n-1}}{1 - \rho} . \tag{12}$$

Hence $Q_n^{-1} P_n$ is Cauchy and so converges, so that (1) converges. That $\frac{1}{4}$ is the best possible follows from the scalar case.

It would be desirable to extend the theorem to the case $\sup a_n \leqslant \frac{1}{4}$ so as to get the complete analog of Worpitzky's theorem, but it appears that it will require a proof of a different type than that above.

Reference

1. J. Worpitzky, "Untersuchungen über die Entwickelung der Monodronen und Monogenen Functionen durch Kettenbrüche, "pp. 3–39, Friedrichs-Gymnasium und Realschule, Jahresbericht, Berlin, 1865.